Geometry of Fourfolds with an Admissible K3 Subcategory

Laura Pertusi
Dipartimento di Matematica F. Enriques, Università degli Studi di Milano
Advisor: Paolo Stellari

Setting

A **K3 category** is a triangulated category whose Serre functor is the shift $(-)[2]$ and with the same Hochschild cohomology of a K3 surface. Cubic fourfolds and Gushel-Mukai fourfolds have a semiorthogonal decomposition of their derived category of coherent sheaves given by exceptional objects and an admissible K3 category. This allows us to study:

- Fourier-Mukai partners of cubic fourfolds;
- the double EPW sextic associated to a GM fourfold as a moduli space of twisted sheaves on a K3 surface.

Motivation: K3 categories simplify the study of moduli problems over cubic or GM fourfolds.

Fourier-Mukai partners of cubic fourfolds

A cubic fourfold X is a smooth cubic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.
Semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{O}_X(1), \mathcal{O}_X(2) \rangle$$

where $\mathcal{A}_X$ is a K3 category (Kuznetsov).
Definition: A cubic fourfold X' is a **FM partner** of X if there is an equivalence $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ of Fourier-Mukai type, i.e. there exists $K \in \mathrm{D}^{\mathrm{b}}(X \times X')$ such that

$$\Phi : \mathrm{D}^{\mathrm{b}}(X) \to \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'} \to \mathrm{D}^{\mathrm{b}}(X')$$

$$\Phi(-) \cong \mathrm{R}p_{X'*}(K \overset{\mathrm{L}}{\otimes} \mathrm{L}p_X^*(-)).$$

Consistently with the analogy to K3 surfaces we have:
Theorem:([1]) The number of isomorphism classes of FM partners $\#\mathrm{FM}(X)$ of X is finite.

Question

Are there examples of cubic fourfolds with a prescribed number of non isomorphic FM partners?

Answer: Consider **general** cubic fourfolds of discriminant d with a **Hodge-associated (twisted) K3 surface** (S, α) $\Leftrightarrow$ numerical condition on the discriminant.

Untwisted case (Hassett)

$$4 \nmid d, 9 \nmid d, p \nmid d \ \forall \text{ prime } p \equiv 2 (\mathrm{mod}\, 3) \qquad (*)$$

Twisted case (Huybrechts)

$$n_i \equiv 0 (\mathrm{mod}\, 2) \ \forall \text{ prime } p_i \equiv 2 (\mathrm{mod}\, 3) \text{ in } 2d = \Pi p_i^{n_i} \quad (*')$$

Theorem 1

Let $d > 6$, $d \equiv 0, 2 \,\mathrm{mod}(6)$ satisfying $(*')$ and let h be the number of distinct odd primes in prime factorization of $d/\mathrm{ord}(\alpha)$. Let X be a general element in $\mathcal{C}_d$.
Untwisted case: If d satisfies $(*)$, then

$$\#\mathrm{FM}(X) = \begin{cases} 2^{h-1}, & \text{if } d \equiv 2 (\mathrm{mod}\, 6) \text{ and } h > 1; \\ 2^{h-2}, & \text{if } d \equiv 0 (\mathrm{mod}\, 6) \text{ and } h > 2; \\ 1, & \text{otherwise.} \end{cases}$$

Twisted case: We get a lower bound to $\#\mathrm{FM}(X)$, depending on h and $\varphi(\alpha)$.

Tool: Mukai lattice for $\mathcal{A}_X$

$$\tilde{H}(\mathcal{A}_X, \mathbb{Z}) = \{\kappa \in K_{\mathsf{top}}(X) : \chi([\mathcal{O}_X(i)], \kappa) = 0, \forall i = 0, 1, 2\}.$$

Theorem:([1]) For general $X \in \mathcal{C}_d$, $\mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_{X'}$ of FM type iff their Mukai lattices are Hodge isometric.

Gushel-Mukai fourfolds

A GM fourfold is a smooth dimensionally transverse intersection

$$X = \mathrm{CG}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

where Q is a quadric hypersurface in $\mathbb{P}(W) \cong \mathbb{P}^8 \subset \mathbb{P}(\wedge^2 V_5 \oplus \mathbb{C})$.
Semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(X) = \langle \mathcal{A}_X, \mathcal{O}_X, \mathcal{U}_X^*, \mathcal{O}_X(1), \mathcal{U}_X^*(1) \rangle$$

where $\mathcal{A}_X$ is a K3 category (Kuznetsov, Perry).
Lagrangian data: X defines a triple (V_6, V_5, A), where $A \subset \wedge^3 V_6$ is Lagrangian without decomposable vectors. Viceversa, it is possible to recover the GM fourfold from such a data (Debarre, Kuznetsov).
$\rightsquigarrow$ EPW stratification in $Y_A^{\geq 3} \subset Y_A^{\geq 2} \subset Y_A^{\geq 1} \subset \mathbb{P}(V_6)$ and **EPW sextic** hypersurface $Y_A := Y_A^{\geq 1}$.

Associated double EPW sextic

We consider the double cover of the EPW sextic Y_A branched over $Y_A^{\geq 2}$ associated to a GM fourfold X. Assume that $\tilde{Y}_A$ is smooth $\Leftrightarrow Y_A^{\geq 3} = \emptyset$.

Aim

To study $\tilde{Y}_A$ as a moduli space of (twisted) stable sheaves on a K3 surface.

Facts (Debarre, Iliev, Manivel):

- Period points of **special** GM fourfolds form divisors in the period domain identified by the discriminant d.
- Hodge-associated K3 surface $\Leftrightarrow 8 \nmid d$ and the only odd primes which divide d are $\equiv 1 (\mathrm{mod}\, 4)$ $\qquad (\dagger)$

Assume that X has discriminant d.

Theorem 2 (untwisted case)

If d satisfies $(\dagger)$, then $\tilde{Y}_A$ is birational to a moduli space of stable sheaves on a K3 surface S.
The converse holds for general X and for non general X whose period point is in a divisor with discriminant $d \equiv 2$ or $4 (\mathrm{mod}\, 8)$.

Remark: There are examples of GM fourfolds with $\mathrm{rank}(H^{2,2}(X, \mathbb{Z})) = 4$ and period point only in divisors with discriminants $\equiv 0 (\mathrm{mod}\, 8)$, having $\tilde{Y}_A$ birational to a moduli space of sheaves on a K3 surface, **but** which cannot have a Hodge-associated K3 surface.

Steps:

- Mukai lattice $\tilde{H}(\mathcal{A}_X, \mathbb{Z})$
$$\langle \lambda_1, \lambda_2 \rangle^{\perp} \cong H^4(X, \mathbb{Z})_{\mathrm{van}}$$
- Relate condition $(\dagger)$ with existence of a primitively embedded $U = (\mathbb{Z}^2, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})$ in the algebraic part of $\tilde{H}(\mathcal{A}_X, \mathbb{Z})$.
- There is a primitive embedding of $H^2(\tilde{Y}_A, \mathbb{Z})$ in $\tilde{H}(\mathcal{A}_X, \mathbb{Z}) \rightsquigarrow$ we apply Addington's result.

Theorem 2 (twisted case)

There is a Hodge isometry $\tilde{H}(\mathcal{A}_X, \mathbb{Z}) \cong \tilde{H}(S, \alpha, \mathbb{Z})$ where (S, α) is a twisted K3 surface iff

$$d = \prod_i p_i^{n_i} \text{ with } n_i \equiv 0 (\mathrm{mod}\, 2) \text{ for } p_i \equiv 3 (\mathrm{mod}\, 4) \ (\dagger')$$

$\tilde{Y}_A$ is birational to a moduli space of twisted stable sheaves on a K3 surface S if and only if d satisfies $(\dagger')$.

Theorem 3

$\tilde{Y}_A$ is birational to the Hilbert scheme $S^{[2]}$ on a K3 surface S iff d satisfies

$$a^2 d = 2n^2 + 2 \quad \text{for } a, n \in \mathbb{Z}.$$

Stability conditions on $\mathcal{A}_X$ (joint with X. Zhao, work in progress)

Property:([2]) If X is an ordinary GM fourfold, then the restriction to a hyperplane $\mathbb{P}(V_4) \subset \mathbb{P}(V_5)$ of the first conic fibration ρ is flat and smooth.

$$\begin{array}{ccccc} & & \tilde{X} & \longrightarrow & \mathbb{P}_X(\mathcal{U}_X) \\ & \overset{\sigma}{\swarrow} & \downarrow \tilde{\rho} & & \downarrow \rho \\ X & \longleftarrow & \mathbb{P}(V_4) & \longrightarrow & \mathbb{P}(V_5) \end{array}$$

$\tilde{X} = \mathrm{Bl}_E(X), \quad E = \mathrm{G}(2, V_4) \cap Q.$
Idea: Use $\tilde{\rho}$ to induce stability conditions on $\mathcal{A}_X$ from $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, $\mathcal{B}_0$= even part of associated Clifford algebra.

References

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